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UNPUBLISHED PRELIMINARY DATA

RADIATION FROM HARMONIC SOURCES
IN A
UNIFORMLY MOVING MEDIUM

by

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Department of ELECTRICAL ENGINEERING

THE OHIO STATE UNIVERSITY
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Investigation of   Spacecraft Antenna Problems

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Uniformly Moving Medium

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RADIATION FROM HARMONIC SOURCES IN
A UNIFORMLY MOVING MEDIUM

I. INTRODUCTION

The problem of the electrodynamics of moving media was first solved exactly by Minkowski[1] in 1908, and an excellent discussion of his work has been given by Sommerfeld[2]. Recently, a review of Minkowski's theory and a discussion of several current writings on this subject have been given by Tai[3].

In this report, Minkowski's results are used to find a general integral for the field equations in a moving medium. The medium is assumed to be homogeneous, isotropic, and to move with a constant velocity much less than the speed of light, inside a certain volume of space. Only time harmonic fields are considered. A wave equation for the electric field is derived and is integrated by means of a Green's Identity and an appropriately defined Dyadic Green's Function. The result gives the electric field inside the volume in terms of known sources in the volume and the tangential components of the electric and magnetic fields over the enclosing surface. Finally, the fields radiated by a point dipole in an infinite moving medium are found.

II. INTEGRATION OF THE FIELD EQUATIONS

Consider a homogeneous and isotropic medium of permittivity ϵ and permeability μ. Assume the medium is moving with constant velocity with respect to an inertial xyz-coordinate system. Under the condition that the velocity of the medium is much less than the velocity of light, the electromagnetic fields, as measured in the xyz-frame, are governed by the equations[4]

(1) $$\nabla \times \overline{E} = -\frac{\partial}{\partial t} \left[\mu \overline{H} - (\epsilon\mu - \epsilon_o\mu_o)\overline{v} \times \overline{E} \right]$$

(2) $$\nabla \times \overline{H} = \frac{\partial}{\partial t} \left[\epsilon \overline{E} + (\epsilon\mu - \epsilon_o\mu_o)\overline{v} \times \overline{H} \right] + \overline{J}$$

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where

$$\overline{E}, \overline{H} = \text{the electric and magnetic fields,}$$
$$\epsilon_o, \mu_o = \text{the permittivity and permeability of free-space,}$$

and

$$\overline{J} = \text{the source volume current density.}$$

We consider in this report only the case where all field quantities vary sinusoidally with time. The factor $e^{+j\omega t}$ may then be suppressed and Eqs. (1) and (2) take the form:

(3) $$\nabla \times \overline{E} = - j\omega \left[\mu \overline{H} - (\epsilon\mu - \epsilon_o\mu_o)\overline{v} \times \overline{E}\right]$$

(4) $$\nabla \times \overline{H} = j\omega \left[\epsilon \overline{E} + (\epsilon\mu - \epsilon_o\mu_o)\overline{v} \times \overline{H}\right] + \overline{J}$$

where ω is the radian frequency. Also, we define the vector $\overline{\Lambda}$, [*]

(5) $$\overline{\Lambda} = (\epsilon\mu - \epsilon_o\mu_o) \overline{v}$$

and write Eqs. (3) and (4) as

(6) $$(\nabla - j\omega\overline{\Lambda}) \times \overline{E} = - j\omega\mu \overline{H}$$

(7) $$(\nabla - j\omega\overline{\Lambda}) \times \overline{H} = j\omega\epsilon \overline{E} + \overline{J} .$$

Combining Eqs. (6) and (7) yields the following wave equation for $\overline{E}$

(8) $$(\nabla - j\omega\overline{\Lambda}) \times (\nabla - j\omega\overline{\Lambda}) \times \overline{E} - k^2 \overline{E} = -j\omega\mu \overline{J}$$

where $k = \omega\sqrt{\mu\epsilon}$. Assuming $\overline{J}$ to be known, a solution for $\overline{E}$ is sought.

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[*] We choose the symbol $\overline{\Lambda}$ (an upside-down "V") because this quantity has dimensions of reciprocal velocity.

Equation (8) may be integrated by the use of a Dyadic Green's Function defined by the equation

$$(9) \quad (\nabla + j\omega\overline{\Lambda}) \times (\nabla + j\omega\overline{\Lambda}) \times \overline{\overline{T}}(\overline{R}|\overline{R}') - k^2 \overline{\overline{T}}(\overline{R}|\overline{R}') = \overline{\overline{\epsilon}}\, \delta(|\overline{R} - \overline{R}'|)$$

where $\overline{\overline{\epsilon}}$ is the idemfactor and $\delta(|\overline{R} - \overline{R}'|)$ is the Dirac Delta Function. It may be remarked that the components of $\overline{\overline{T}}$ correspond to the fields resulting from a source radiating in a medium moving with velocity $-\overline{v}$. A solution for Eq. (9) is obtained below.

Equation (9) may be used to solve Eq. (8) by means of a suitable Green's Identity, as follows. Let $\overline{P}$ and $\overline{Q}$ be two vector fields with continuous second derivatives at all points of a certain volume of space V. Define the vector $\overline{A}$ as

$$(10) \quad \overline{A} = \overline{P} \times (\nabla - j\omega\overline{\Lambda}) \times \overline{Q} - \overline{Q} \times (\nabla + j\omega\overline{\Lambda}) \times \overline{P}.$$

Then it is easily verified that

$$(11) \quad \nabla \cdot \overline{A} = \overline{Q} \cdot (\nabla + j\omega\overline{\Lambda}) \times (\nabla + j\omega\overline{\Lambda}) \times \overline{P} - \overline{P} \cdot (\nabla - j\omega\overline{\Lambda}) \times (\nabla - j\omega\overline{\Lambda}) \times \overline{Q}$$

and hence from the Divergence Theorem applied to the volume V

$$(12) \quad \iiint\limits_{V} [\overline{Q} \cdot (\nabla + j\omega\overline{\Lambda}) \times (\nabla + j\omega\overline{\Lambda}) \times \overline{P} - \overline{P} \cdot (\nabla - j\omega\overline{\Lambda}) \times (\nabla - j\omega\overline{\Lambda}) \times \overline{Q}]\, dv$$

$$= \oiint\limits_{S} [\overline{P} \times (\nabla - j\omega\overline{\Lambda}) \times \overline{Q} - \overline{Q} \times (\nabla + j\omega\overline{\Lambda}) \times \overline{P}] \cdot \hat{n}\, dS$$

where S is a regular surface bounding the volume V and $\hat{n}$ is an outward directed normal to S. By arranging Eq. (12) so $\overline{P}$ appears last in each combination of vectors and setting

$$(13) \quad \overline{P} = \overline{\overline{T}}(\overline{R}|\overline{R}') \cdot \overline{a}$$

(14) $\overline{Q} = \overline{E}$

where $\overline{a}$ is an arbitrary constant vector, it is found that $\overline{a}$ may be re-
moved from the equation and the following identity for the vector-
dyadic mixture results:

(15) $$\iiint_V \{\overline{E} \cdot (\nabla+j\omega\overline{\Lambda}) \times (\nabla+j\omega\overline{\Lambda}) \times \overline{\overline{T}} - [(\nabla-j\omega\overline{\Lambda}) \times (\nabla-j\omega\overline{\Lambda}) \times \overline{E}] \cdot \overline{\overline{T}}\} dv$$

$$= \oiint_S \hat{n} \cdot \{[(\nabla-j\omega\overline{\Lambda}) \times \overline{E}] \times \overline{\overline{T}} - \overline{E} \times [(\nabla+j\omega\overline{\Lambda}) \times \overline{\overline{T}}]\} dS.$$

Finally, substituting Eqs. (6), (8) and (9) yields the general solution
for E

(16) $$\overline{E}(\overline{R}') = -j\omega\mu \iiint_V \overline{J}(\overline{R}) \cdot \overline{\overline{T}}(\overline{R}|\overline{R}')dv$$

$$- \oiint_S \{-j\omega\mu(\hat{n}\times\overline{H}(\overline{R})) \cdot \overline{\overline{T}}(\overline{R}|\overline{R}') + (\hat{n}\times\overline{E}(\overline{R})) \cdot [(\nabla+j\omega\overline{\Lambda}) \times \overline{\overline{T}}(\overline{R}|\overline{R}')]\} dS$$

by which the electric field inside an arbitrary volume may be calculated
from the sources within the volume and the tangential fields on the sur-
face. Equation (16) is a mathematical statement of Huygen's Principle
as applied to moving media.

III. THE DYADIC GREEN'S FUNCTION

A direct method of obtaining the solution for Eq. (9) is to make
the assumption that $\overline{\overline{T}}$ may be written as the product of a scalar function
ϕ and another dyadic $\overline{\overline{\Gamma}}$

(17) $\overline{\overline{T}} = \phi\overline{\overline{\Gamma}}$

where ϕ is to be chosen to simplify Eq. (9). From the identity

(18) $(\nabla + j\omega\overline{\Lambda}) \times \phi\overline{\overline{\Gamma}} = \phi\nabla \times \overline{\overline{\Gamma}} + (\nabla\phi + j\omega\overline{\Lambda}\phi) \times \overline{\overline{\Gamma}}$

it is seen that if ϕ is chosen to satisfy the differential equation

(19) $\nabla\phi + j\omega\overline{\Lambda}\phi = 0$

then Eq. (18) gives the simple formula

(20) $(\nabla + j\omega\overline{\Lambda}) \times \phi\overline{\overline{\Gamma}} = \phi\nabla\times\overline{\overline{\Gamma}}$

and also

(21) $(\nabla + j\omega\overline{\Lambda}) \times (\nabla + j\omega\overline{\Lambda}) \times \phi\overline{\overline{\Gamma}} = \phi\nabla\times\nabla\times\overline{\overline{\Gamma}}$

Hence for such a ϕ, Eq. (9) reduces to

(22) $\phi[\nabla\times\nabla\times\overline{\overline{\Gamma}} - k^2\overline{\overline{\Gamma}}] = \overline{\overline{\epsilon}}\,\delta(|\overline{R} - \overline{R}'|)$

which has the same form as the equation for the free-space Dyadic Green's Function.

When the velocity of the medium is zero, the dyadic $\overline{\overline{T}}$ must reduce to the free-space Dyadic Green's Function (with the propagation constant $k = \omega\sqrt{\mu\epsilon}$ replacing the free-space propagation constant $k_0 = \omega\sqrt{\mu_0\epsilon_0}$). Hence it may be seen from Eq. (22) that when the velocity is zero, ϕ must be unity. Therefore, the solution to Eq. (19) is

(23) $\phi = e^{-j\omega\overline{\Lambda}\cdot\overline{R}}$

and Eq. (22) becomes

(24) $\nabla\times\nabla\times\overline{\overline{\Gamma}} - k^2\overline{\overline{\Gamma}} = e^{+j\omega\overline{\Lambda}\cdot\overline{R}}\overline{\overline{\epsilon}}\,\delta(|\overline{R} - \overline{R}'|) = e^{+j\omega\overline{\Lambda}\cdot\overline{R}'}\overline{\overline{\epsilon}}\,\delta(|\overline{R} - \overline{R}'|)$.

The last equality in Eq. (24) follows from the property of the Dirac Delta Function. The solution for $\overline{\overline{\Gamma}}$ may be written down at once from the known solution for the free-space Dyadic Green's Function,

$$(25) \qquad \overline{\overline{\Gamma}}(\overline{R}|\overline{R}') = \frac{1}{4\pi} \, e^{+j\omega\overline{\Lambda}\cdot\overline{R}'} \left[\overline{\overline{\epsilon}} - \frac{\nabla\nabla'}{k^2}\right] \frac{e^{-jk|\overline{R}-\overline{R}'|}}{|\overline{R}-\overline{R}'|}$$

and then from Eq. (17) the solution for $\overline{\overline{T}}(\overline{R}|\overline{R}')$ is

$$(26) \qquad \overline{\overline{T}}(\overline{R}|\overline{R}') = \frac{1}{4\pi} \, e^{j\omega\overline{\Lambda}\cdot(\overline{R}-\overline{R}')} \left[\overline{\overline{\epsilon}} - \frac{\nabla\nabla'}{k^2}\right] \frac{e^{-jk|\overline{R}-\overline{R}'|}}{|\overline{R}-\overline{R}'|} \quad .$$

As mentioned above, $\overline{\overline{T}}(\overline{R}|\overline{R}')$ reduces to the free-space Dyadic Green's Function when the velocity of the medium is zero.

Equations (16) and (26) comprise the general solution for the radiation of harmonic sources in a uniformly moving medium. Although it is perhaps obvious, it is worth pointing out that in using Eq. (26) in Eq. (16), one must pay careful attention to the primed and unprimed coordinates, since the sign of the phase constant $e^{j\omega\overline{\Lambda}\cdot(\overline{R}-\overline{R}')}$ changes when the roles of $\overline{R}$ and $\overline{R}'$ are interchanged. In the next section, the radiation properties of an infinitesimal electric dipole are examined.

IV. THE INFINITESIMAL ELECTRIC DIPOLE

Consider an electric dipole of vanishingly small dimensions and of dipole moment p_0. With the dipole located at the coordinate origin and oriented in the z-direction, the source current density is

$$(27) \qquad \overline{J}(\overline{R}') = j\omega p_0 \, \delta(|\overline{R}'-0|) \, \hat{z} \, .$$

Assuming the dipole radiates into an infinite medium, only the volume integral in Eq. (16) is needed. For this source, the volume integral is easily done and gives for the θ-component of the electric field

(28) $\qquad E_\theta = \dfrac{-1}{4\pi}\, \omega^2 \mu\, p_0 \sin\theta\, e^{+j\omega\overline{\Lambda}\cdot\overline{R}}\, \dfrac{e^{-jkR}}{R}\left[1 - \dfrac{j}{kR} + \dfrac{1}{(kR)^2}\right]$

which reduces to

(29) $\qquad E_\theta = -\dfrac{1}{4\pi}\, \omega^2 \mu p_0\, e^{+j\omega\overline{\Lambda}\cdot\overline{R}} \sin\theta\, \dfrac{e^{-jkR}}{R}$

in the far-field. With no loss of generality, we may let

(30) $\qquad \overline{\Lambda} = \Lambda_x\, \hat{x} + \Lambda_z\, \hat{z}$

and then substituting for the unit vectors gives

(31) $\qquad \overline{\Lambda}\cdot\overline{R} = (\Lambda_x \sin\theta \cos\phi + \Lambda_z \cos\theta)\, R\,.$

The magnetic field associated with Eq. (29) may be calculated from

(32) $\qquad \overline{H} = \dfrac{-1}{j\omega\mu_o}\, (\nabla - j\omega\overline{\Lambda})\times\overline{E}\,.$

When this expression is applied to Eq. (29) it is found that the radial components of $\nabla\times\overline{E}$ and $-j\omega\overline{\Lambda}\times\overline{E}$ cancel out, leaving only a ϕ-component of $\overline{H}$ given by

(33) $\qquad H_\phi = \sqrt{\dfrac{\epsilon}{\mu}}\; E_\theta = \dfrac{1}{\eta}\, E_\theta$

where η is the characteristic impedance of the medium.

Thus it is seen that the effect of the velocity of the medium is to change the propagation constant in the medium, because of the factor $e^{+j\omega\overline{\Lambda}\cdot\overline{R}}$. Since Λ is a constant vector, the phase constant for the medium will be larger than k in one-half the space and smaller in the other half. To compute the total power radiated by the dipole, the quantity $1/2\, E_\theta H_\phi{}^*$ (the asterick denotes the complex conjugate) may be integrated

over the surface of a shpere. However, since the phase term $e^{+j\omega\bar{\Lambda}\cdot\bar{R}}$ will not appear in the product $E_\theta H_\phi{}^*$, it is seen that the total power radiated by the dipole and hence its radiation resistance are unaffected by the velocity of the medium. Also. the radiation "pattern" of the dipole does not depend on the velocity.

V. CONCLUSIONS

The wave equation for the electric field in a uniformly moving medium has been derived and a Green's Identity suitable for integrating the wave equation has been found. Also, the equation for the appropriate Dyadic Green's Function has been solved. The form of the Green's Function was found to differ from that of the free-space Green's Function only by an additional phase factor $e^{-j\omega\bar{\Lambda}\cdot(\bar{R}-\bar{R}')}$. The integral in Eq. (16) along with Eq. (26) for the Dyadic Green's Function give the general solution for the electric field inside a volume of space in terms of sources within the volume and tangential fields on the surface of the volume.

Also, the radiation properties of a point dipole were examined. It was found that the radiated power, radiation resistance, and "pattern" of the dipole were all unaffected by the motion of the medium. Only the phase velocity was affected, its value being larger than the corresponding value for a stationary medium in one-half of space and smaller in the other half.

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